HW 8

1. Show that $\mathcal{L}(E)$ (consisting of all measurable functions on $E \in \mathfrak{M}$ with "norm"

$$\|f\| := \int_E |f| \quad \forall f \in \mathcal{L}(E))$$

is a normed (vector) space, i.e.

$$\|\lambda f\| = |\lambda| \cdot \|f\| \quad \forall \lambda \in \mathbb{R}, \forall f \in \mathcal{L}(E).$$

$$\|f_1 + f_2\| \leq \|f_1\| + \|f_2\| \quad \forall f_1, f_2 \in \mathcal{L}(E).$$

$0 \leq \|f\|$ with equality iff $f = 0$

(equivalent functions are identified).

$\overset{*}{2}$. Let $E \in \mathfrak{M}$, and $L_\infty(E)$ consists of all $f \in \mathfrak{MF}(E)$ such that

$$\|f\|_\infty := \inf\{M \in [0,\infty) : |f(x)| \leq M \text{ a.e } x \text{ in } E\}$$

(equivalent functions are identified). Show that "inf" can be replaced by "min" in the above expression of $\|f\|_\infty$. Show that $L_\infty(E)$ is a Banach space, i.e. it is a normed vector space and each Cauchy sequence $(f_n)_{n \in \mathbb{N}}$ in $L_\infty(E)$ has a "limit" f in $L_\infty(E)$, i.e.

if $\forall\, \varepsilon > 0 \ \exists\, N$ s.t

$$\|f_m - f_n\|_\infty < \varepsilon \quad \forall\, m, n \geq N.$$

then $\exists\, f \in L_\infty(E)$ s.t. $\forall\, \varepsilon > 0 \ \exists\, N \in \mathbb{N}$ s.t.

$$\|f_n - f\|_\infty < \varepsilon \quad \forall\, n \geq N.$$

(Hint: Let $A_n, B_{m,n} \in \mathcal{M}_0$ s.t

$|f_n| \leq \|f_n\|_\infty$ on $E \setminus A_n$, and $|f_m - f_n| \leq \|f_m - f_n\|_\infty$ on $E \setminus B_{m,n}$

Then $A := \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup \left(\bigcup_{m,n\in\mathbb{N}} B_{m,n}\right) \in \mathcal{M}_0$ and

$\left(f_n(x)\right)_{n\in\mathbb{N}}$ converges with limit denoted by

$f(x)$, $\forall x \in E \setminus A$.)

3. Let $(V, \|\cdot\|)$ be a normed vector space and $(v_n)_{n\in\mathbb{N}}$ Cauchy in V. Let (v_{n_k}) be a convergent subsequence of (v_n) convergent to v. Show that (v_n) converges to v.

(Hint. Let $\varepsilon > 0$ and take $N, K \in \mathbb{N}$ s.t.

$$\|v_m - v_n\| < \varepsilon \quad \forall\ m, n \geq N$$

and

$$\|v_{n_k} - v\| < \varepsilon \quad \forall\ k \geq K.$$

Consider m with the form n_k and apply the triangle-inequality).

4.* Let $(V, \|\cdot\|)$ be of the property that if $v_n \in V$ $\forall n \in \mathbb{N}$ with

$$\sum_{n=1}^{\infty} \|v_n\| < +\infty$$

then $v := \sum_{n=1}^{\infty} v_n$ exists in V, i.e.

$$\lim_{N} \|v - \sum_{n=1}^{N} v_n\| = 0$$

Show that $(V, \|\cdot\|)$ is a Banach space, that is, any Cauchy sequence (v_n) in V converges.

Hint. Take a strictly increasing sequence (N_k) of natural nos. $N_1 < N_2 < N_3 \cdots$

$$\|v_m - v_n\| < \frac{1}{2^k}, \ \forall\, m, n \geq N_k$$

Then

$$v_{N_1} + (v_{N_2} - v_{N_1}) + \cdots + (v_{N_m} - v_{N_{m-1}})$$

is a convergent subseq. of $(v_n)_{n\in\mathbb{N}}$.

Apply Q3.

Q5*. Let $m(E)<+\infty$ and $0\leq f\in\mathcal{L}(E)$; and, $\forall n\in\mathbb{N}$,

$$A_n := \{x\in E : n\leq f(x)\}\cap[-n,n]$$

$$B_{n,k} := \left\{x\in E : \frac{k-1}{2^n}\leq f(x)<\frac{k}{2^n}\right\}\cap[-n,n] \quad (k=1,2,\cdots,n\cdot 2^n)$$

and $$\varphi_n = n\chi_{A_n} + \sum_{k=1}^{n\cdot 2^n}\frac{k-1}{2^n}\chi_{B_{n,k}}\in\mathcal{S}_0(E).$$

Show that $0\leq\varphi_n\uparrow_n f$ and $\forall\,\varepsilon>0$

$\exists$ $\begin{cases}\text{simple function } \varphi \\ \text{step function } \psi \\ \text{cts function } g\end{cases}$ vanishing outside a finite-length interval $[a,b]$

s.t. $$\int_E|\varphi-f|,\ \int_E|\psi-f|,\ \int_E|f-g|<\varepsilon.$$